Roy–Steiner equations for πN scattering

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(↪ see also following talk by M. Hoferichter on [JHEP **1206** (2012) 063])

Outline

Motivation: Why πN scattering? Why Roy–Steiner equations?

- Renewed interest in πN **scattering**:
  - $\pi N \to \pi N$ amplitudes e.g. for **σ-term** physics
  - $\bar{N}N \to \pi\pi$ crossed amplitudes e.g. for **nucleon form factors**

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Roy(–Steiner) equations $=$ **P**artial-**W**ave (**H**yberbolic) **D**ispersion **R**elations
coupled by **unitarity** and **crossing** symmetry

- **PW(H)DR**s together with unitarity, crossing symmetry, and chiral symmetry

$\Rightarrow$ Can study processes at low energies with **high precision**:
  - $\pi\pi$ scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011)]

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➢ **Roy–Steiner equations for πN scattering**:
  - Obtain low-energy (pseudophysical) amplitudes with better precision (update input & give errors)
  - Framework allows for systematic improvements (subtractions, higher partial waves, ...)

Warm-up: Roy equations for $\pi\pi$ scattering (1)

- $\pi\pi \to \pi\pi$ is **fully crossing symmetric** in Mandelstam variables s, t, and $u = 4M_\pi^2 - s - t$
- **Roy equations** respect all available symmetry constraints:

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- **Roy equations** respect all available symmetry constraints:

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- Start from twice-subtracted **fixed-t DR**s of the generic form $\hookrightarrow s + t + u = 4M_\pi^2 = s' + t + u'$

$$T(s,t) = c(t) + \frac{1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'^2} \left\{ \frac{s^2}{s'-s} + \frac{u^2}{s'-u} \right\} \operatorname{Im} T(s',t)$$

- Determine subtraction functions $c(t)$ via crossing symmetry
- **PW** expansion ($I \in \{0,1,2\}$, $J = \ell$): $T^I(s,t) = 32\pi \sum_{J=0}^{\infty} (2J+1) P_J\big(\cos\theta(s,t)\big)\, t_J^I(s)$
- PW decomposition of these DRs yields the **Roy equations** [Roy (1971)]

$$t_J^I(s) = k_J^I(s) + \frac{1}{\pi} \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4M_\pi^2}^{\infty} \mathrm{d}s'\, K_{JJ'}^{II'}(s,s') \operatorname{Im} t_{J'}^{I'}(s')$$

- Kernels: analytically known, contain Cauchy kernel $K_{JJ'}^{II'}(s,s') = \frac{\delta^{II'}\delta_{JJ'}}{s'-s} + \ldots$

Warm-up: Roy equations for $\pi\pi$ scattering (2)

$$t_J^I(s) = k_J^I(s) + \frac{1}{\pi}\sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4M_\pi^2}^{\infty} \mathrm{d}s'\, K_{JJ'}^{II'}(s,s')\, \mathrm{Im}\, t_{J'}^{I'}(s')$$

- **Validity**: $4\,M_\pi^2 \leq s \leq 60\,M_\pi^2 \approx (1.08\,\text{GeV})^2$ $\hookrightarrow$ Mandelstam analyticity $\Rightarrow s \leq 68\,M_\pi^2 \approx (1.15\,\text{GeV})^2$
- Subtraction constants (free parameters) contained in $k_J^I(s)$: $\pi\pi$ **scattering lengths**

  $\Rightarrow$ Matching to **Ch**iral **P**erturbation **T**heory [Colangelo et al. (2001)]

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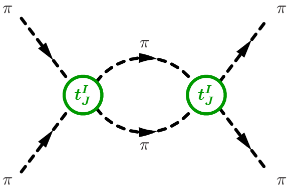
- **Elastic unitarity** leads to coupled integral equations for the **phase shifts** $\delta_J^I(s)$

$$\mathrm{Im}\,t_J^I(s) = \sigma^\pi(s)\left|t_J^I(s)\right|^2\theta\left(t-4M_\pi^2\right)$$

$$\Rightarrow \quad \sigma^\pi(s)\,t_J^I(s) = \frac{e^{2i\delta_J^I(s)}-1}{2i} = \sin\delta_J^I(s)\,e^{i\delta_J^I(s)}$$

$$\sigma^\pi(s) = \sqrt{1-\frac{4M_\pi^2}{s}}$$



πN scattering basics

- Generically: $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$
- Kinematics:

  $s = (p+q)^2\,, \quad t = (p-p')^2\,, \quad u = (p-q')^2$

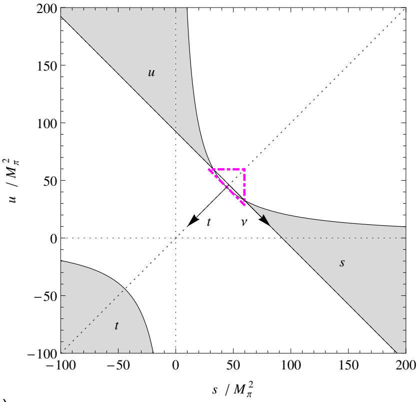
  $u = 2(m^2 + M_\pi^2) - s - t\,, \qquad \nu = \frac{s-u}{4m}$
- **Isospin** structure:

  $T^{ba} = \delta^{ba} T^+ + i\epsilon^{bac}\tau^c T^-$
- **Lorentz** structure ($I \in \{+,-\}$):

  $T^I = \bar{u}(p')\left\{A^I + \frac{\not{q}'+\not{q}}{2} B^I\right\} u(p)$
- **Crossing** symmetry relates amplitudes for

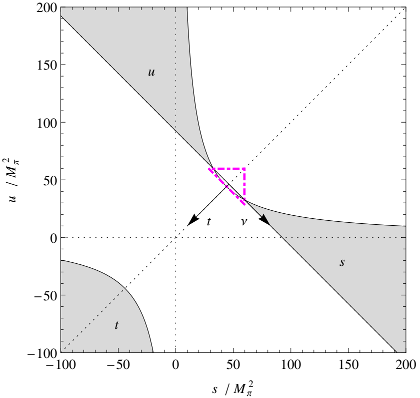
  s-/u-channel ($\pi N \to \pi N$) and t-channel ($\bar{N}N \to \pi\pi$),

  crossing **even** and **odd** amplitudes: $A^\pm(\nu,t) = \pm A^\pm(-\nu,t)\,, \qquad B^\pm(\nu,t) = \mp B^\pm(-\nu,t)$



πN scattering basics: Subthreshold expansion

- Subtraction of pseudovector Born terms: $X \mapsto \bar{X}$
- $D^{\pm} = A^{\pm} + \nu B^{\pm}$
- **Expand** crossing **even** amplitudes
  $X^I(\nu^2, t) \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\}$
  around subthreshold point $\nu = t = 0$:
  $X^I(\nu^2, t) = \sum_{m,n=0}^{\infty} x^I_{mn} (\nu^2)^m t^n$



πN scattering basics: Partial Waves

- **PW**s allow for easy incorporation of **unitarity** constraints $\hookrightarrow$ helicity formalism [Jacob/Wick (1959)]

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- s-channel PW projection: $z_s = \cos\theta_s$, $W = \sqrt{s}$

$$\mathcal{A}^I_\ell(s) = \int_{-1}^{1} \mathrm{d}z_s \, P_\ell(z_s)\mathcal{A}^I(s,t)\big|_{t=t(s,z_s)}$$

$$f^I_{\ell\pm}(W) = \frac{1}{16\pi W}\Big\{(E+m)\big[A^I_\ell(s) + (W-m)B^I_\ell(s)\big] + (E-m)\big[-A^I_{\ell\pm1}(s) + (W+m)B^I_{\ell\pm1}(s)\big]\Big\}$$

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$$A^I(s,t)\big|_{s=s(t,z_t)} = -\frac{4\pi}{p_t^2}\sum_J (2J+1)(p_t q_t)^J \Big\{P_J(z_t) f^J_+(t) - \frac{m}{\sqrt{J(J+1)}} z_t P_J(z_t) f^J_-(t)\Big\}$$

$$B^I(s,t)\big|_{s=s(t,z_t)} = 4\pi \sum_{J>0} \frac{2J+1}{\sqrt{J(J+1)}} (p_t q_t)^{J-1} P_J(z_t) f^J_-(t)$$

- **G-parity** $\Rightarrow$ even J for $I=+$ $(I_t=0)$, odd J for $I=-$ $(I_t=1)$

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- s-channel PW expansion and t-channel PW projection in analogy

Roy–Steiner equations for πN scattering: Hyperbolic DRs

- (Unsubtracted) **Hyperbolic DRs**: $\hookrightarrow (s-a)(u-a) = b = (s'-a)(u'-a)$ with $a, b \in \mathbb{R} \;\Rightarrow\; b = b(s,t,a)$

$$A^+(s,t) = \frac{1}{\pi}\int\limits_{(m+M_\pi)^2}^{\infty} ds' \left[\frac{1}{s'-s} + \frac{1}{s'-u} - \frac{1}{s'-a}\right] \operatorname{Im} A^+(s',t') + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} dt' \,\frac{\operatorname{Im} A^+(s',t')}{t'-t}$$

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$$N^+(s,t) = g^2\left[\frac{1}{m^2-s} - \frac{1}{m^2-u}\right]$$

and similarly for A^-, B^-, N^- [Hite/Steiner (1973)]

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- Why **HDR**s?
  - Combine **all** physical regions $\hookrightarrow$ important for reliable continuation to the subthreshold region [Stahov (1999)]
  - Imaginary parts are only needed in regions where the corresponding PW decompositions converge
  - **Range of convergence** can be maximized by tuning the free hyperbola parameter a
  - Especially powerful for the determination of the **σ-term** [Koch (1982)]

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- How to derive **closed R**oy–**S**teiner **system** of **PWHDR**s:
  1. **Expand** s-/t-channel imaginary parts of **HDR**s in s-/t-channel **PW**s, respectively
  2. **Project** nucleon pole terms and all imaginary parts onto both s- and t-channel **PW**s
  3. **Combine** resulting **RS equations** with the s- & t-channel (extended) **PW unitarity relations**

Roy–Steiner equations for πN scattering: s-channel RS equations

- s-channel PW projection of pole terms and s-/t-channel-PW-expanded imaginary parts

  $\Rightarrow$ (unsubtracted) s-channel **RS equations**:

$$f^I_{\ell+}(W) = N^I_{\ell+}(W) + \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t' \sum_J \Big\{ G_{\ell J}(W,t')\,\mathrm{Im}\,f^J_+(t') + H_{\ell J}(W,t')\,\mathrm{Im}\,f^J_-(t') \Big\}$$

$$+ \frac{1}{\pi}\int\limits_{m+M_\pi}^{\infty} \mathrm{d}W' \sum_{\ell'=0}^{\infty} \Big\{ K^I_{\ell\ell'}(W,W')\,\mathrm{Im}\,f^I_{\ell'+}(W') + K^I_{\ell\ell'}(W,-W')\,\mathrm{Im}\,f^I_{(\ell'+1)-}(W') \Big\}$$

$$= -f^I_{(\ell+1)-}(-W) \quad \forall\, \ell \geq 0$$

[Hite/Steiner (1973)]

- Kernels: analytically known, e.g. $K^I_{\ell\ell'}(W,W') = \frac{\delta_{\ell\ell'}}{W'-W} + \ldots$

- **Validity**: $\hookrightarrow$ above threshold, assuming Mandelstam analyticity $a = -23.19\,M_\pi^2 \quad \Rightarrow$

  $s \in \big[(m+M_\pi)^2 = 59.64\,M_\pi^2, 97.30\,M_\pi^2\big] \quad \Leftrightarrow \quad W \in \big[m+M_\pi = 1.08\,\text{GeV}, 1.38\,\text{GeV}\big]$

Roy–Steiner equations for πN scattering: t-channel RS equations

- t-channel PW projection of pole terms and s-/t-channel-PW-expanded imaginary parts

  $\Rightarrow$ (unsubtracted) t-channel **RS equations**:

$$f_+^{J\geq 0}(t) = \tilde{N}_+^J(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t' \sum_{J'} \Big\{ \tilde{K}_{JJ'}^1(t,t')\,\mathrm{Im} f_+^{J'}(t') + \tilde{K}_{JJ'}^2(t,t')\,\mathrm{Im} f_-^{J'}(t') \Big\}$$

$$+ \frac{1}{\pi}\int_{m+M_\pi}^{\infty} \mathrm{d}W' \sum_{\ell=0}^{\infty} \Big\{ \tilde{G}_{J\ell}(t,W')\,\mathrm{Im} f_{\ell+}^I(W') + \tilde{G}_{J\ell}(t,-W')\,\mathrm{Im} f_{(\ell+1)-}^I(W') \Big\}$$

$$f_-^{J\geq 1}(t) = \tilde{N}_-^J(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}t' \sum_{J'>0} \tilde{K}_{JJ'}^3(t,t')\,\mathrm{Im} f_-^{J'}(t')$$

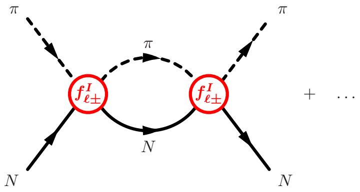
$$+ \frac{1}{\pi}\int_{m+M_\pi}^{\infty} \mathrm{d}W' \sum_{\ell=0}^{\infty} \Big\{ \tilde{H}_{J\ell}(t,W')\,\mathrm{Im} f_{\ell+}^I(W') + \tilde{H}_{J\ell}(t,-W')\,\mathrm{Im} f_{(\ell+1)-}^I(W') \Big\}$$

- Kernels analytically known, e.g. $\tilde{K}_{JJ'}^1(t,t') = \frac{\delta_{JJ'}}{t'-t} + \ldots\,, \qquad \tilde{K}_{JJ'}^3(t,t') = \frac{\delta_{JJ'}}{t'-t} + \ldots$
- **Validity**: $\hookrightarrow$ above pseudothreshold, assuming Mandelstam analyticity $a = -2.71\,M_\pi^2 \quad \Rightarrow$

  $t \in \big[4M_\pi^2, 205.45\,M_\pi^2\big] \quad \Leftrightarrow \quad \sqrt{t} \in \big[2M_\pi = 0.28\,\text{GeV}, 2.00\,\text{GeV}\big]$

- s-channel **unitarity relations** ($I_s \in \{1/2, 3/2\}$):

$$\text{Im}\, f_{\ell\pm}^{I_s}(W) = q_s \left| f_{\ell\pm}^{I_s}(W) \right|^2 \theta\big(W-(m+M_\pi)\big) + \frac{1-\left[\eta_{\ell\pm}^{I_s}(W)\right]^2}{4q_s}\,\theta\big(W-(m+2M_\pi)\big)$$



Roy–Steiner equations for πN scattering: Unitarity relations

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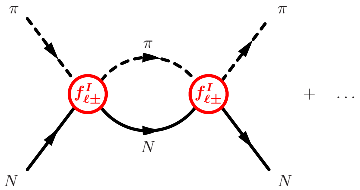
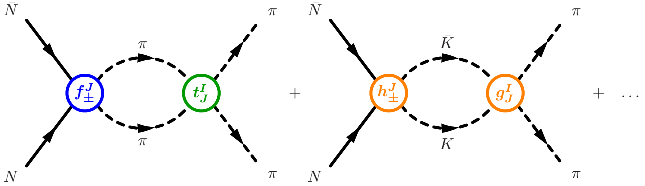


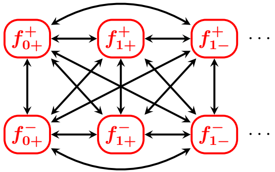
- t-channel **(extended) unitarity relations**: ↪ (2-body intermediate states: $\pi\pi$ & $\bar{K}K$ + ...)

$$\text{Im}\, f^J_\pm(t) = \sigma^\pi_t \left(t^I_J(t)\right)^* f^J_\pm(t)\, \theta\left(t-4M_\pi^2\right) + c_J 2\sqrt{2}\, k_t^{2J} \sigma^K_t \left(g^I_J(t)\right)^* h^J_\pm(t)\, \theta\left(t-4M_K^2\right) + \ldots$$



  - Only **linear** in $f^J_\pm(t)$ ⇒ less restrictive
  - **Watson's theorem**: $\arg f^J_\pm(t) = \delta^I_J(t)$ [Watson (1954)] ↪ for $t < 16\,M_\pi^2 \lesssim 40\,M_\pi^2 \approx (0.88\,\text{GeV})^2$

- s-channel **subproblem**:
  - Kernels are diagonal for $I \in \{+,-\}$, but unitarity relations are diagonal for $I_s \in \{1/2, 3/2\}$
    $\Rightarrow$ **All** PWs are interrelated
  - Once the t-channel PWs are known
    $\Rightarrow$ Structure similar to $\pi\pi$ **Roy equations**



Roy–Steiner equations for πN scattering: Recoupling schemes

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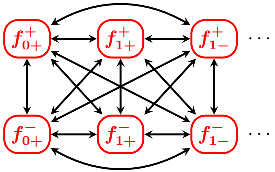
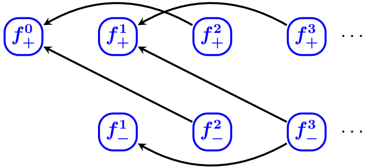
  - Once the t-channel PWs are known
    $\Rightarrow$ Structure similar to $\pi\pi$ **Roy equations**



- t-channel **subproblem**:
  - Only higher PWs couple to lower ones
  - Only PWs with even or odd J are coupled
  - No contribution from $f_+^{J'}$ to f_-^J
  - $\Rightarrow$ Leads to **Muskhelishvili–Omnès problem**



Roy–Steiner equations for πN scattering: t-channel subproblem (1)

- Linear combinations $\quad \Gamma^J(t) = m\sqrt{\frac{J}{J+1}}f_-^J(t) - f_+^J(t) \quad \forall J \geq 1$
- (unsubtracted) t-channel **subproblem** can be written as

$$f_+^0(t) = \Delta_+^0(t) + \frac{t-4m^2}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im} f_+^0(t')}{(t'-4m^2)(t'-t)} \qquad \left[f_+^0(4m^2)=0\right]$$

$$\Gamma^{J\geq 1}(t) = \Delta_\Gamma^J(t) + \frac{t-4m^2}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im}\,\Gamma^J(t')}{(t'-4m^2)(t'-t)} \qquad \left[\Gamma^J(4m^2)=0\right]$$

$$f_-^{J\geq 1}(t) = \Delta_-^J(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\text{Im} f_-^J(t')}{t'-t}$$

with $\quad \text{Im} f_\pm^J(t) = \sigma_t^\pi \left(t_J^I(t)\right)^* f_\pm^J(t)\,\theta\left(t-4M_\pi^2\right) + \dots$

- Inhomogeneities $\Delta(t)$: Born terms, s-channel integrals, and higher t-channel PWs; e.g.

$$\Delta_-^J(t) = \tilde{N}_-^J(t) + \frac{1}{\pi}\int_{m+M_\pi}^{\infty} dW' \sum_{\ell=0}^{\infty}\left\{\tilde{H}_{J\ell}(t,W')\,\text{Im} f_{\ell+}^I(W') + \tilde{H}_{J\ell}(t,-W')\,\text{Im} f_{(\ell+1)-}^I(W')\right\}$$
$$+\frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \sum_{J'\geq J+2} \tilde{K}_{JJ'}^3(t,t')\,\text{Im} f_-^{J'}(t')$$

Roy–Steiner equations for πN scattering: t-channel subproblem (2)

- In the **low-energy** (pseudophysical) region:
  - Only the lowest s-/t-channel PWs are relevant
  - Can match amplitudes to **ChPT** [Büttiker/Meißner (2000), Becher/Leutwyler (2001), ...]
  - Neglect inelasticities in both the $\pi\pi$- and the t-channel PWs $\hookrightarrow \eta_J^I(t) = 1$ & no $\bar{K}K + \ldots$

  $\Rightarrow$ **Watsons's theorem**, single-channel approximation of t-channel **subproblem**

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➢ (Single-channel) **M**uskhelishvili–**O**mnès **problem** with **finite matching point** t_m

[Muskhelishvili (1953), Omnès (1958), Büttiker et al. (2004)]

$$f(t) = \Delta(t) + \frac{1}{\pi}\int_{4M_\pi^2}^{t_\mathrm{m}} \mathrm{d}t' \frac{\sin\delta(t')e^{-i\delta(t')}f(t')}{t'-t} + \frac{1}{\pi}\int_{t_\mathrm{m}}^{\infty} \mathrm{d}t' \frac{\mathrm{Im}\, f(t')}{t'-t} \equiv |f(t)|e^{i\delta(t)} \quad \text{for } t \leq t_\mathrm{m} < t_\mathrm{inel}$$

- Solving for $|f(t)|$ in $4M_\pi^2 \leq t \leq t_\mathrm{m}$ requires: $\delta(t)$ for $4M_\pi^2 \leq t \leq t_\mathrm{m}$ & $\mathrm{Im}\, f(t)$ for $t \geq t_\mathrm{m}$
- Solution via once-subtracted **Omnès function** with $t_\mathrm{m} < \infty$ $\hookrightarrow \Omega(0) = 1$

$$\Omega(t) = \exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^{t_\mathrm{m}} \frac{\mathrm{d}t'}{t'}\frac{\delta(t')}{t'-t}\right\} = \exp\left\{\frac{t}{\pi}\fint_{4M_\pi^2}^{t_\mathrm{m}} \frac{\mathrm{d}t'}{t'}\frac{\delta(t')}{t'-t}\right\} e^{i\delta(t)\,\theta(t-4M_\pi^2)\theta(t_\mathrm{m}-t)}$$

Roy–Steiner equations for πN scattering: Subtractions

- In general: Subtractions
  - May be necessary to ensure the **convergence** of DR/MO integrals $\hookrightarrow$ asymptotic behavior
  - Can be introduced to lessen the dependence of the low-energy solution on the high-energy behavior
  - Parametrize high-energy information in (a priori unknown) subtraction constants $\hookrightarrow$ matching to ChPT

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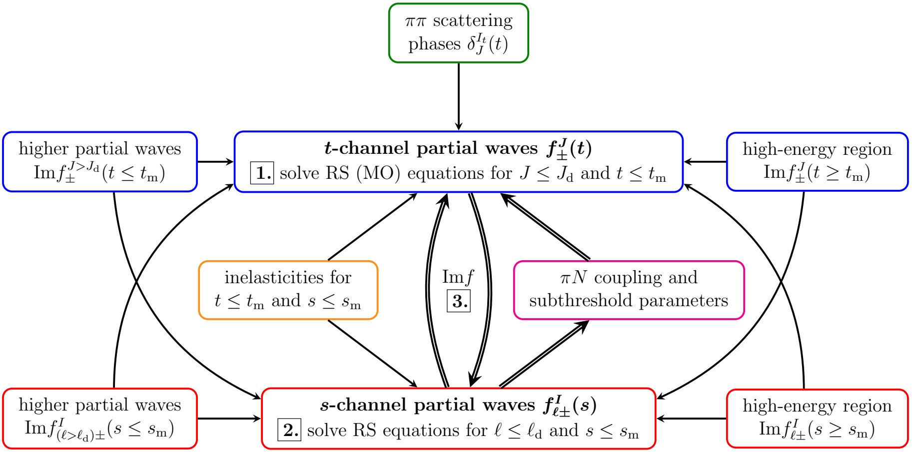
  - Parametrize high-energy information in (a priori unknown) subtraction constants $\hookrightarrow$ matching to ChPT
- Favorable choice for t-channel **MO problem**: subthreshold expansion around $\nu = t = 0$
  - **Subtract HDR**s for $A^\pm$ and $B^\pm$ at $s = u = m^2 + M_\pi^2$ and $t = 0$
  - Done up to full second order; added (partial) third subtraction for $A^\pm$
  - $\Rightarrow$ Obtain **sum rules** for subthreshold parameters x_{mn}^I
  - $\Rightarrow$ General structure of RS/MO problem remains unchanged
- **HDR**s $\Rightarrow$ s-/t-channel **RS equations** (pole terms & kernels) $\Rightarrow$ t-channel **MO problem**,

  e.g. for **P-waves** ($n \geq 1$):

$$\Gamma^1(t) = \Delta_\Gamma^1(t)\Big|^{n\text{-sub}} + \frac{t^{n-1}(t-4m^2)}{\pi} \int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t' \, \frac{\operatorname{Im}\Gamma^1(t')}{t'^{n-1}(t'-4m^2)(t'-t)}$$

$$f_-^1(t) = \Delta_-^1(t)\Big|^{n\text{-sub}} + \frac{t^n}{\pi} \int\limits_{4M_\pi^2}^{\infty} \mathrm{d}t' \, \frac{\operatorname{Im} f_-^1(t')}{t'^n(t'-t)}$$

Roy–Steiner equations for πN scattering: Solution strategy



t-channel Muskhelishvili–Omnès problem: Input

- Here, show results for the ***P*-waves**, since
  - ***S*-wave**: Strong effect from $\bar{K}K$ intermediate states ($f_0(980)$ resonance)

    $\Rightarrow$ need two-channel MO analysis $\Rightarrow$ **following talk**
  - ***P*-waves**: Single-channel MO approximation well justified in the low-energy region
  - D-waves: Dominated by nucleon pole terms $\hookrightarrow$ in general for all PWs for $t \to 4M_\pi^2$
- First step: Check **consistency** with KH80 t-channel PWs $\hookrightarrow$ iteration with s-channel results t.b.d.

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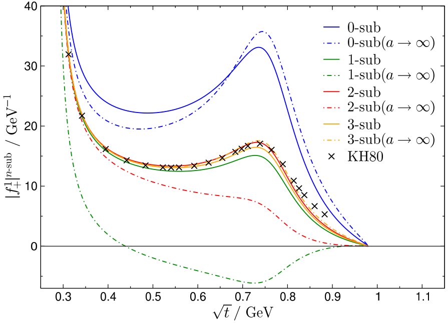
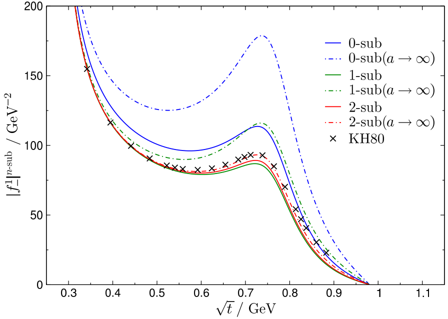
- **Input used**:
  - $\pi\pi$ phase shifts δ_J^I [Caprini/Colangelo/Leutwyler (in preparation)]
  - s-channel: SAID PWs [Arndt et al. (2008)] for $W \leq 2.5$ GeV, above: Regge model [Huang et al. (2010)]
  - KH80 [Höhler (1983)] subthreshold parameters & **coupling** $g^2/(4\pi) = 14.28$

    $\hookrightarrow$ modern value: $g^2/(4\pi) = 13.7 \pm 0.2$ [Baru et al. (2011)]
  - t-channel: **All** contributions above $t_\mathrm{m} = \mathbf{0.98}$ **GeV** set to **zero** $\Rightarrow$ solutions fixed $f_J^I(t_\mathrm{m}) = 0$

- f_+^J less well determined in MO framework than f_-^J, since
  - Effectively one subtraction less $\Rightarrow$ introduced partial third subtraction
  - Enhanced sensitivity to subtraction constants $\hookrightarrow \tilde{N}_+^0(4M_\pi^2) = \tilde{N}_\Gamma^J(4M_\pi^2) = 0$
- Estimate systematic **uncertainties** (1): "fixed-t limit" $|a| \to \infty$ $\hookrightarrow$ modulo t-channel integrals





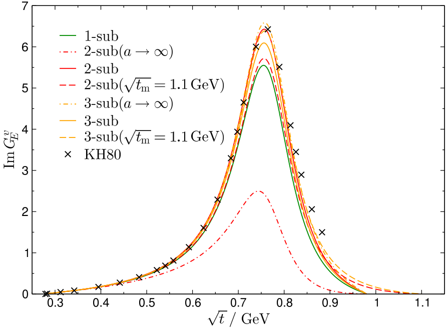
- Estimate systematic **uncertainties** (2): Variation of the matching point t_m $\Rightarrow$ similar...

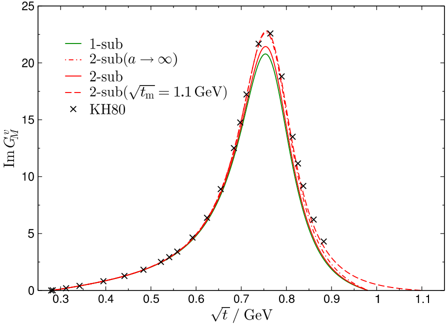
➢ MO solutions in general consistent with KH80 results

t-channel Muskhelishvili–Omnès problem: Isovector spectral functions

- ***P*-waves** feature in dispersive analyses of the **Sachs form factors** of the nucleon:

$$\mathrm{Im}\, G_E^v(t) = \frac{q_t^3}{m\sqrt{t}} \big(F_\pi^V(t)\big)^* f_+^1(t)\, \theta\big(t-4M_\pi^2\big) \qquad \mathrm{Im}\, G_M^v(t) = \frac{q_t^3}{\sqrt{2t}} \big(F_\pi^V(t)\big)^* f_-^1(t)\, \theta\big(t-4M_\pi^2\big)$$





Summary & Outlook

- What has been done:
  - Derived a closed system of **Roy–Steiner equations** (PWHDRS) for πN **scattering**
  - Constructed **unitarity relations** including $\bar{K}K$ intermediate states for the t-channel PWs
  - Optimized the **range of convergence** by tuning a for s- and t-channel each
  - Implemented **subtractions** at several orders
  - **Solved** the t-channel (single-channel) **MO problem**

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  ➢ t-channel RS/MO machinery works $\hookrightarrow$ modulo the S-wave
- What needs to be done:
  - Two-channel MO analysis for the S-wave, effect on scalar form factor $\Rightarrow$ **following talk**
  - Numerical solution of the s-channel **subproblem** using the t-channel results as input

  ➢ Self-consistent, iterative solution of the **full RS system** $\Rightarrow$ lowest PWs & low-energy parameters
  - Possible improvements: Higher subtractions, higher PWs, more inelastic input, ...

πN scattering basics

- Generically: $\pi^a(q) + N(p) \to \pi^b(q') + N(p')$
- Kinematics:

$$s = (p+q)^2\,, \quad t = (p-p')^2\,, \quad u = (p-q')^2$$

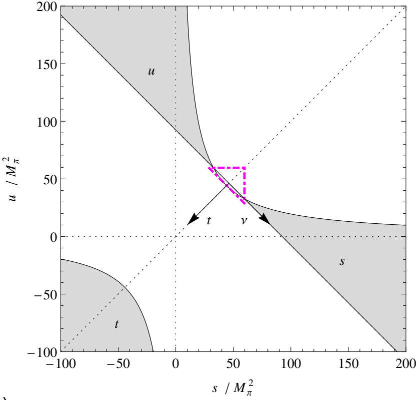
$$u = 2(m^2 + M_\pi^2) - s - t\,, \qquad \nu = \frac{s-u}{4m}$$

- **Isospin** structure:

$$T^{ba} = \delta^{ba} T^+ + i\epsilon^{bac}\tau^c T^-$$

- **Lorentz** structure ($I \in \{+,-\}$):

$$T^I = \bar{u}(p')\left\{A^I + \frac{\not{q}' + \not{q}}{2} B^I\right\} u(p)$$



- **Crossing** symmetry relates amplitudes for **s-/u-channel** ($\pi N \to \pi N$) and **t-channel** ($\bar{N}N \to \pi\pi$), crossing **even** and **odd** amplitudes: $A^\pm(\nu,t) = \pm A^\pm(-\nu,t)\,, \qquad B^\pm(\nu,t) = \mp B^\pm(-\nu,t)$
- **Isospin & crossing** $\Rightarrow$ $\begin{pmatrix} \mathcal{A}^{I=+} \\ \mathcal{A}^{I=-} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \mathcal{A}^{I_s=1/2} \\ \mathcal{A}^{I_s=3/2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \mathcal{A}^{I_t=0} \\ \mathcal{A}^{I_t=1} \end{pmatrix}, \quad \mathcal{A} \in \{A, B\}$

πN scattering basics: Subthreshold expansion

- Subtraction of pseudovector Born terms: $X \mapsto \bar{X}$
- $D^{\pm} = A^{\pm} + \nu B^{\pm}$
- Expand crossing even amplitudes

  $X^I(\nu^2, t) \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\}$

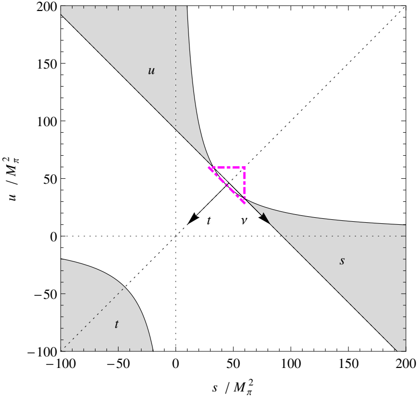
  around subthreshold point $\nu = t = 0$:

  $X^I(\nu^2, t) = \sum_{m,n=0}^{\infty} x^I_{mn} (\nu^2)^m t^n$

- Relations between subthreshold parameters x^I_{mn}:

  $d^+_{mn} = a^+_{mn} + b^+_{m-1,n} \quad \Rightarrow \quad d^+_{0n} = a^+_{0n}$

  $d^-_{mn} = a^-_{mn} + b^-_{mn}$



- Subthreshold expansion of $A^{\pm}$ and $B^{\pm}$:

$$A^+(\nu, t) = \frac{g^2}{m} + d^+_{00} + d^+_{01} t + a^+_{10} \nu^2 + \mathcal{O}\left(\nu^4, \nu^2 t, t^2\right)$$

$$A^-(\nu, t) = a^-_{00} \nu + a^-_{01} \nu t + \mathcal{O}\left(\nu^3, \nu t^2\right) \qquad B^+(\nu, t) = \frac{g^2}{M_\pi^2} \frac{4m\nu}{M_\pi^2} + b^+_{00} \nu + \mathcal{O}\left(\nu^3, \nu t\right)$$

$$B^-(\nu, t) = -\frac{g^2}{2m^2} - \frac{g^2}{M_\pi^2} \left[2 + \frac{t}{M_\pi^2} \right] + b^-_{00} + b^-_{01} t + \mathcal{O}\left(\nu^2, \nu^2 t, t^2\right)$$

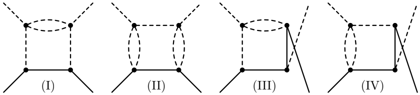
Roy–Steiner equations for πN scattering: Range of convergence

- Assumption: **Mandelstam analyticity** [Mandelstam (1958,1959)]

$$T(s,t) = \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}u' \frac{\rho_{su}(s',u')}{(s'-s)(u'-u)} + \frac{1}{\pi^2}\iint \mathrm{d}t'\mathrm{d}u' \frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}t' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

with integration ranges defined by the support of the **double spectral regions ρ**

- Boundaries of ρ are given by the lowest graphs



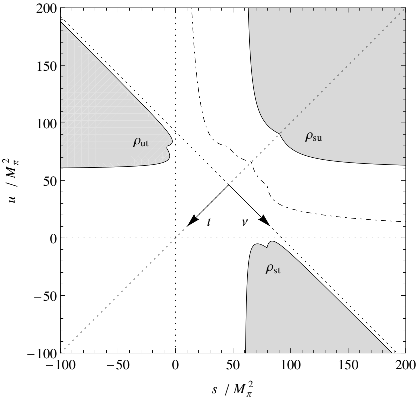
- Convergence of PW exps. of imaginary parts

  $\Rightarrow$ **Lehman ellipses** for $z = \cos\theta$ [Lehmann (1958)]

- Convergence of PW projs. of full equations

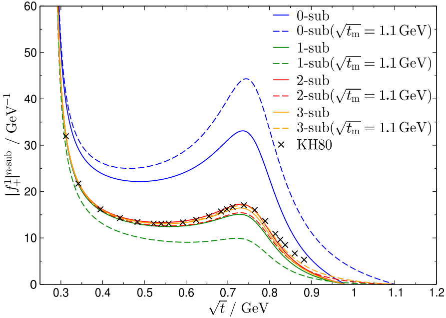
  $\Rightarrow$ for given a, hyperbolas must not enter any ρ for all needed values of b

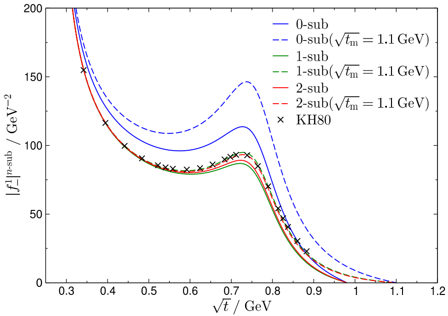
➢ Constraints on b yield ranges in s & t



t-channel Muskhelishvili–Omnès problem: P-waves (2)

- Estimate systematic **uncertainties**: Variation of the matching point t_m $\hookrightarrow$ effect of $f^I_J(t_\mathrm{m}) = 0$





- Convergence pattern & internal consistency ✔
- Consistency with KH80 ✔

➢ MO solutions in general consistent with KH80 results