Unitarity and Analyticity Constraints on π-K Form Factors

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- The process and definitions

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- Based on the papers:

  Gauhar Abbas and BA, European Physical Journal A 41 (2009) 7.

  Gauhar Abbas, BA, I. Caprini, I. Sentitemsu Imsong and S. Ramanan, European Physical Journal A 44 (2010) 175; European Physical Journal A 45 (2010) 389.

  Gauhar Abbas, BA, I. Caprini and I. Sentitemsu Imsong , Physical Review, D 82 (2010) 094018

  Recent updates: arXiv:1112.4270 (PoS(RADCOR2011)036), and arXiv:1202.5399 (DAE-BRNS Workshop proceedings, to appear)

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- Applied also to heavy-light system $D\pi$ form factors: BA, Irienl Caprini and I. Sentitemsu Imsong, European Physical Journal, A 47 (2011) 147.

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$$l^\mu = \overline{u}(p_\nu)\gamma^\mu(1-\gamma_5)v(p_l)$$

$$F^+(p', p)_\mu = \langle \pi^0(p')|\bar{s}\gamma_\mu u|K^+(p)\rangle = \frac{1}{\sqrt{2}}((p'+p)_\mu f_+(t) + (p-p')_\mu f_-(t))$$

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- Neutral $F_\mu^0(p', p)$ defined without the $1/\sqrt{2}$
  Recent review for isospin violation, A. Kastner and H. Neufeld, European Physical Journal C57 (2008) 541.
- $f_+(t), t = (p'-p)^2$ is known as the vector form factor as it is the P-wave projection of the crossed channel matrix element $\langle 0|\bar{s}\gamma_\mu u|K^+\pi^0, \mathrm{in}\rangle$.

Definitions continued

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- Consider the expansion about $t = 0$

$$f_0(t) = f_+(0) \left(1 + \lambda_0' \frac{t}{M_\pi^2} + \frac{1}{2}\lambda_0'' \frac{t^2}{M_\pi^4} + \cdots \right),$$

$\lambda_0' = M_\pi^2 \langle r_{\pi \mathrm{K}}^2 \rangle / 6$, $\lambda_0'' = 2 M_\pi^4\, c$ are related to the radius $\langle r_{\pi \mathrm{K}}^2 \rangle$ and curvature, c used alternatively in the literature.

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- Analogously defined for the vector form factor.

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- More recently from τ decays: BELLE has fitted them with resonances in the time-like region on the unitarity cut.
- Solutions of Muskelishvili-Omnès equations for form factors using phase shift information and some additional inputs to self- consistently generate them. Work of Moussallam, group of Jamin, Oller, Pich, Boito, Escribano.

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- Recent determinations from the lattice, e.g., RBC+UKQCD collaboration [P. A. Boyle et al., Physical Review Letters 100 (2008) 141601] gives $f_+(0) = 0.964(5)$. They use 2+1 flavour of dynamical wall quarks. (recent update, G. Colangelo et al., European Physical Journal, C (2011) 71:1695 [FLAG report] gives 0.956 ± 0.008)

Low energy theorems - I

- A soft-pion theorem due to Callan and Treiman (C. G. Callan and S. B. Treiman, Physical Review Letters 16 (1966) 153) says

$$f_0(M_K^2 - M_\pi^2) = F_K/F_\pi + \Delta_{CT}$$

$\Delta_{CT} \simeq 0$ to two-loops in chiral perturbation theory (J. Bijnens and P. Talavera, Nuclear Physics B 669 (2003) 341.)
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  This point called CT_1 is above the end-point of the K_{l3} but is in the analyticity part of the timelike region.
- Knowledge of F_K/F_π at high precision is therefore crucial.

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  This point known as CT_2 is in the spacelike region.
- Difficult to estimate higher order corrections (to our knowledge not yet done in the literature).

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- An extremely interesting joint analysis of $f_+(0)$ and F_K/F_π is by V. Bernard and E. Passemar, JHEP 1004 (2010) 001

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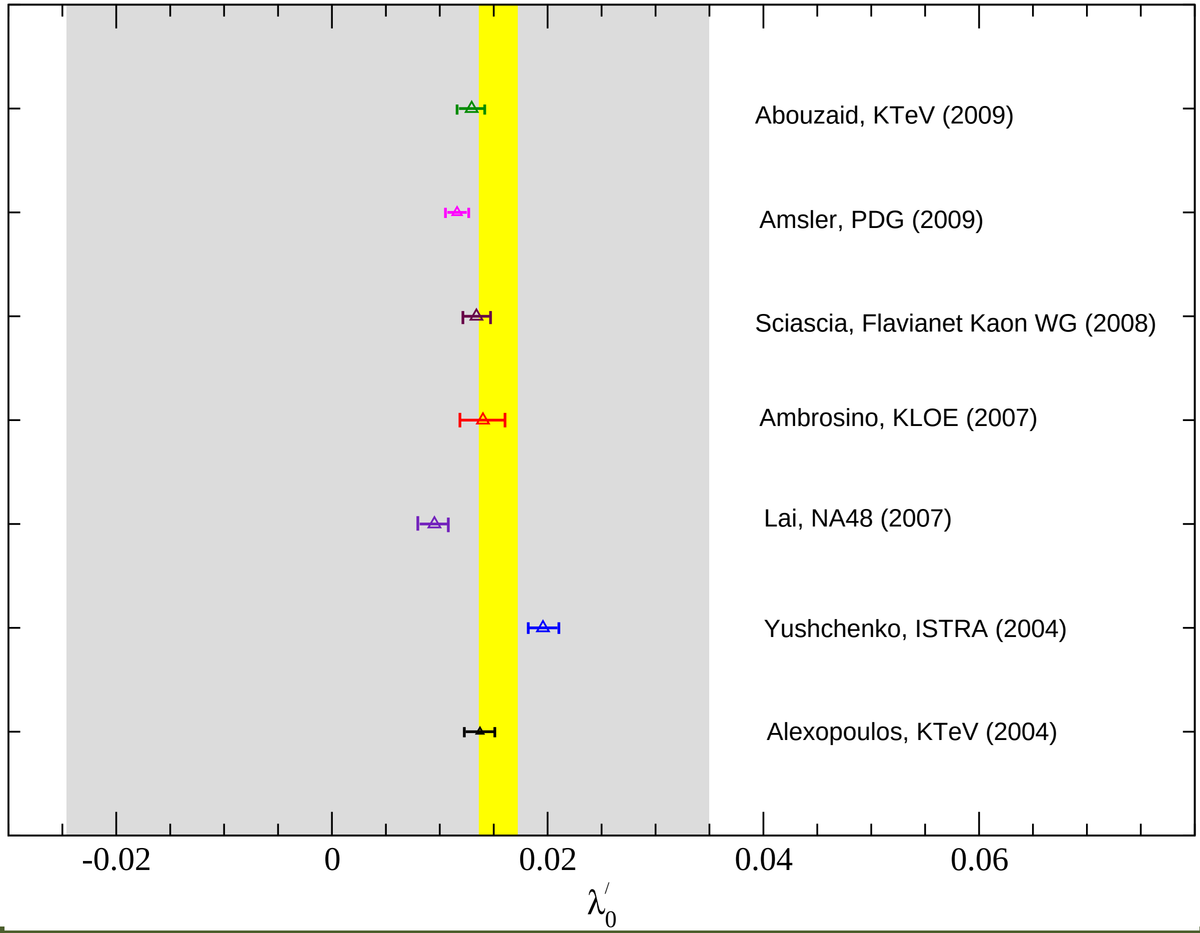
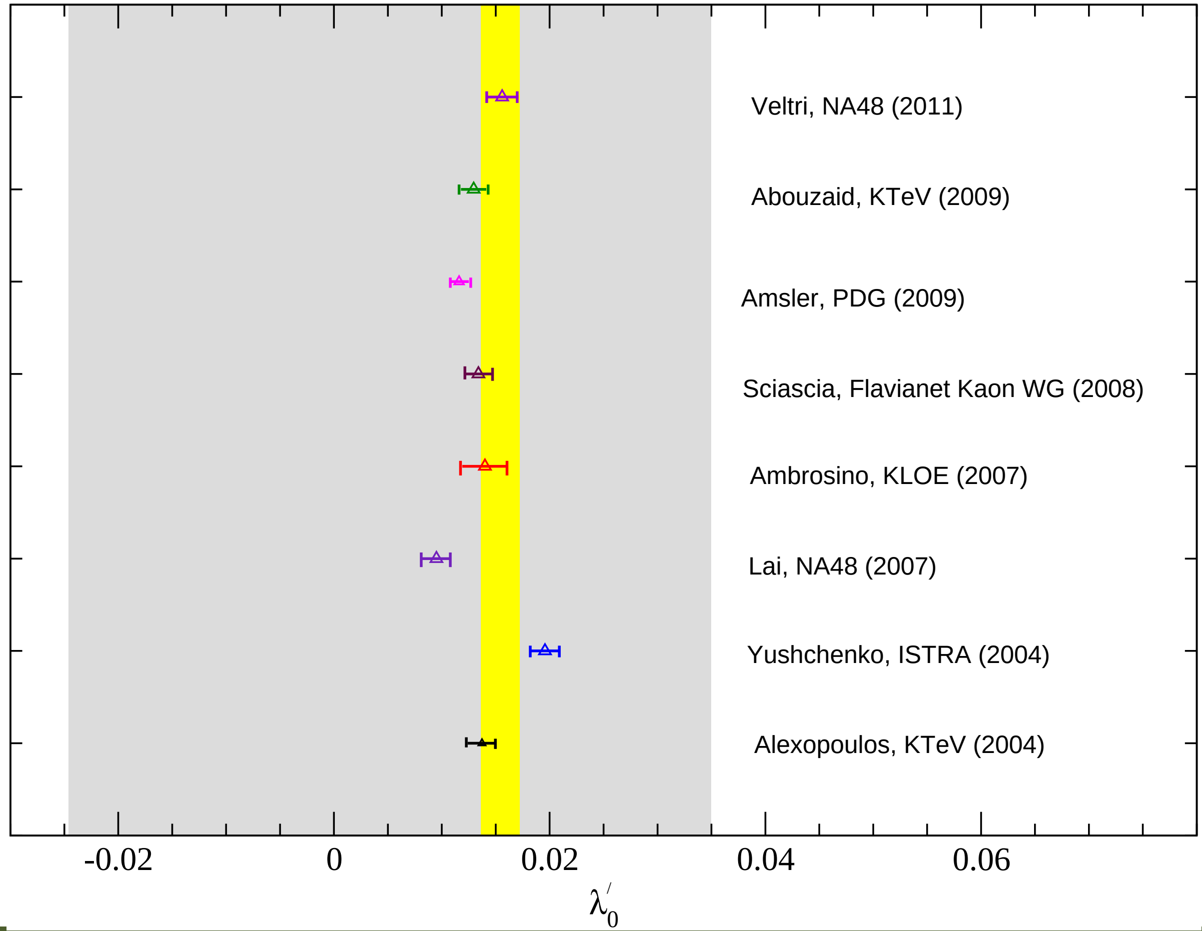
- KTeV experiment at Fermilab. 1.9 million K_L electron and 1.5 million K_L muono decays.
  T. Alexopoulos et al., Physical Review D 70 (2004) 092007
  E. Abouzaid et al., Physical Review D 81 (2010) 052001

Scalar experiments – summary



Updated summary



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- Mushkelishvili-Omnès study of πK, πK^*, $K\rho$ and use of high statistics LASS experiment phase shifts used to produce the πK vector form factor and compared with BELLE (B. Moussallam, European Physical Journal C 53 (2008) 401)

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- Series of studies based on these data: M. Jamin et al. Physics Letters B 664 (2008) 78; B 640 (2006) 176
  D. R. Boito et al., European Physical Journal C 59 (2009) 821.

Theoretical approaches

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- Our phase and modulus data come from Moussallam, group of Jamin et al., and from BELLE.

QCD correlator $\chi_0(Q^2)$ - I

- Consider the QCD correlator

$$\chi_0(Q^2) \equiv \frac{\partial}{\partial q^2}\left[q^2\Pi_0\right] = \frac{1}{\pi}\int_{t_+}^{\infty} dt\, \frac{t\mathrm{Im}\Pi_0(t)}{(t+Q^2)^2}\,,$$

$$\mathrm{Im}\Pi_0(t) \geq \frac{3}{2}\,\frac{t_+t_-}{16\pi}\,\frac{[(t-t_+)(t-t_-)]^{1/2}}{t^3}\,|f_0(t)|^2\,,$$

with $t_\pm = (M_K \pm M_\pi)^2$.

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  with $t_\pm = (M_K \pm M_\pi)^2$.
- Positive definite and can be bounded.
- Bounds can be obtained using analyticity to transform the problem, and to input values of the form factor and its derivatives at $t = 0$ and/or knowledge at various points in the analyticity region (method of unitarity bounds).

QCD correlator $\chi_0(Q^2)$ - II

- On the other hand, in pQCD when $Q \gg \Lambda_{\rm QCD}, m_q, \alpha_S$ $\overline{MS}$ scheme.

$$\chi_0(Q^2) = \frac{3(m_s - m_u)^2}{8\pi^2 Q^2}\left[1 + 1.80\alpha_s + 4.65\alpha_s^2 + 15.0\alpha_s^3 + 57.4\alpha_s^4 \ldots\right].$$

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- Reverse problem: to constrain λ_0', λ_0'' and $f_0(\Delta_{K\pi})$ and $f_0(\overline{\Delta}_{K\pi})$.

Transforming via Conformal map

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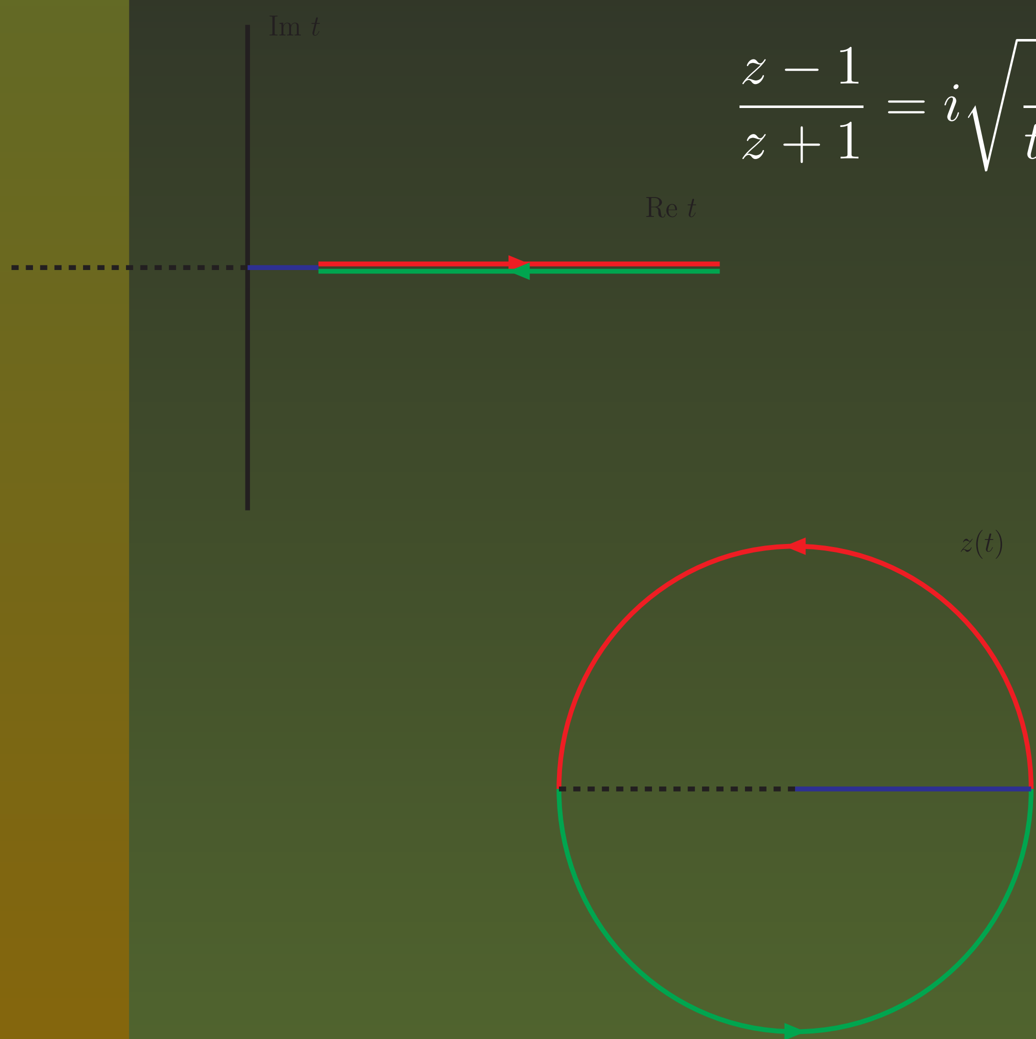

The problem transformed

- We can now use the conformal map to transform this to an integral that reads

$$\frac{1}{2\pi}\int_0^{2\pi} |h(\exp(i\theta))|^2 \leq I_{\rm pQCD}$$

and needs to be bounded.

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- For the case at hand:

$$\begin{aligned} w(z) &= \frac{3}{16\sqrt{2\pi}}\frac{M_K - M_\pi}{M_K + M_\pi}\sqrt{1-z}\,(1+z)^{3/2} \\ &\quad\times \frac{(1+z(-Q^2))^2}{(1-z\,z(-Q^2))^2}\frac{(1-z\,z(t_-))^{1/2}}{(1+z(t_-))^{1/2}}, \\ h(z) &= w(z) f_0(z). \end{aligned}$$

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- Theory of Hardy Spaces (H^2) involves square integrable functions on the open unit disc
- Ideal setting for us since the original integral now is reduced to a series expansion on the Hardy Space and involves only the expansion coefficients.

Power series and origin of the bound

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- Furthermore and significantly, square integrability implies $I = |a_0|^2 + |a_1|^2 + ...$ [Parseval theorem]
- Outer function is known and can be expanded in a series in z.
- If the first n coefficients of the form factor are known, a bound on the quantity of interest is obtained after a finite number of terms.

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- $$\begin{aligned} a_2 &= \frac{h''(0)}{2!} = \frac{f_+(0)}{2}\left[w(0)\left(-\frac{8}{3}\langle r^2_{\pi \mathrm{K}}\rangle t_+ + 32\, c\, t_+^2\right)\right] \\ &+ \frac{f_+(0)}{2}\left[2w'(0)\left(\frac{2}{3}\langle r^2_{\pi \mathrm{K}}\rangle t_\pi\right) + w''(0)\right], \end{aligned}$$

Improving the bounds

- Improvement of the bound arises if $f_0(t)$ is known for some spacelike values of momenta corresponding to $z = x_i,\ i = 1, 2, 3, ...$

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- Improve the bound by using imposing constraints using Lagrange multipliers.
- Can also be improved by imposing phase of the form factor for timelike moment in a continuous region, $a \leq t \leq b$.

Spacelike constraints

- Can be extended to arbitrary number of such constraints, and mixed constraints (Meiman problem). The problem solved in generality by A. Raina and V. Singh, Journal of Physics G3 (1977) 315.

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- The case of two spacelike constraints is one where we solve:

$$\begin{vmatrix} I & a_0 & a_1 & a_2 & J_1 & J_2 \\ a_0 & 1 & 0 & 0 & 1 & 1 \\ a_1 & 0 & 1 & 0 & x_1 & x_2 \\ a_2 & 0 & 0 & 1 & x_1^2 & x_2^2 \\ J_1 & 1 & x_1 & x_1^2 & (1-x_1^2)^{-1} & (1-x_1x_2)^{-1} \\ J_2 & 1 & x_2 & x_2^2 & (1-x_1^2)^{-1} & (1-x_2^2)^{-1} \end{vmatrix} = 0$$

to obtain the bound, if a_i and J_i are known. Here I and J_i are known, and hence we can bound the a_i!

Inclusion of phase and modulus

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- Adaptation of method first proposed by Caprini in 1999 in the context of the pion electromagnetic form factor (I. Caprini, European Physical Journal C 13 (2000) 471).
- The present work is the only other known application of this powerful technique which is described in the following.

Omnès function

- Consider the definition

$$\mathcal{O}(t) = \exp\left(\frac{t}{\pi}\int_{t_+}^{\infty} dt \frac{\delta(t')}{t'(t'-t)}\right),$$

where $\delta(t)$ is the $I = 1/2$ elastic S-wave $K\pi$ scattering phase, in the elastic region and arbitrary Lipschitz continuous above t_{in} (viz., the phase and its first derivative are continuous).

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- Since the Omnès function $\mathcal{O}(t)$ fully accounts for the second Riemann sheet of the form factor, the function $h(t)$, defined by

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is real analytic in the t-plane with a cut only for $t \geq t_{\rm in}$.

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- Extremely clever trick which makes the method very useful

New conformal map

- The new conformal variable is now:

$$z(t) = \frac{\sqrt{t_{\rm in}} - \sqrt{t_{\rm in} - t}}{\sqrt{t_{\rm in}} + \sqrt{t_{\rm in} - t}},$$

which maps the t-plane cut for $t > t_{\rm in}$ onto the unit disk $|z| < 1$, and

$$h(z) = f_0(t(z))\, w(z)\, \omega(z)\, [\mathcal{O}(t(z))]^{-1},$$

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- **Note that the Omnès function makes an appearance through its outer function ($\omega(z)$) and once as an inverse.**

New Outer functions

- The new outer function is

$$w(z) = \frac{3(M_K^2 - M_\pi^2)}{16\sqrt{2\pi}t_{\rm in}} \frac{\sqrt{1-z}\,(1+z)^{3/2}(1+z(-Q^2))^2}{(1-z\,z(-Q^2))^2} \times \frac{(1-z\,z(t_+))^{1/2}\,(1-z\,z(t_-))^{1/2}}{(1+z(t_+))^{1/2}\,(1+z(t_-))^{1/2}},$$

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- The input for the bound is now given by

$$I = \chi_0(Q^2) - \frac{3}{2}\frac{t_+ t_-}{16\pi^2}\int_{t_+}^{t_{\rm in}} dt\, \frac{[(t-t_+)(t-t_-)]^{1/2}|f_0(t)|^2}{t^2(t+Q^2)^2}.$$

Information of the modulus used in the integral.

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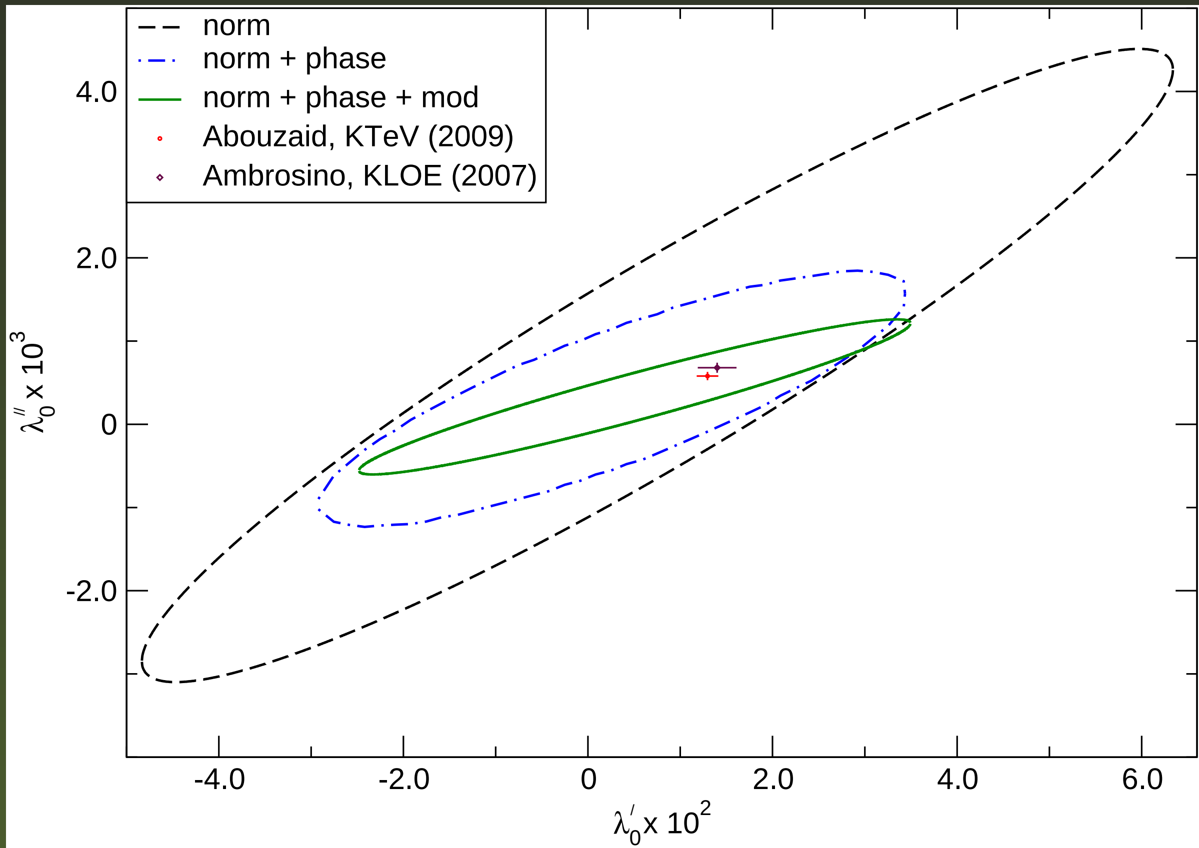
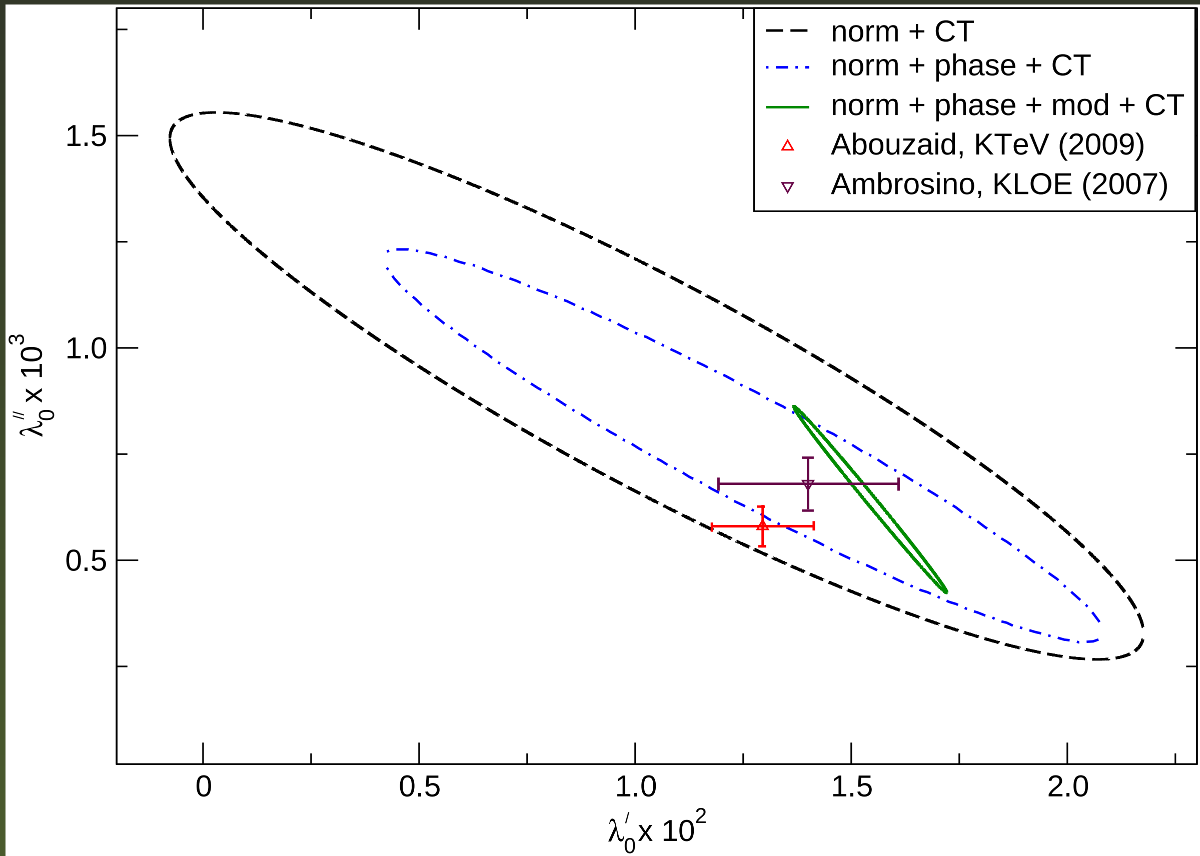
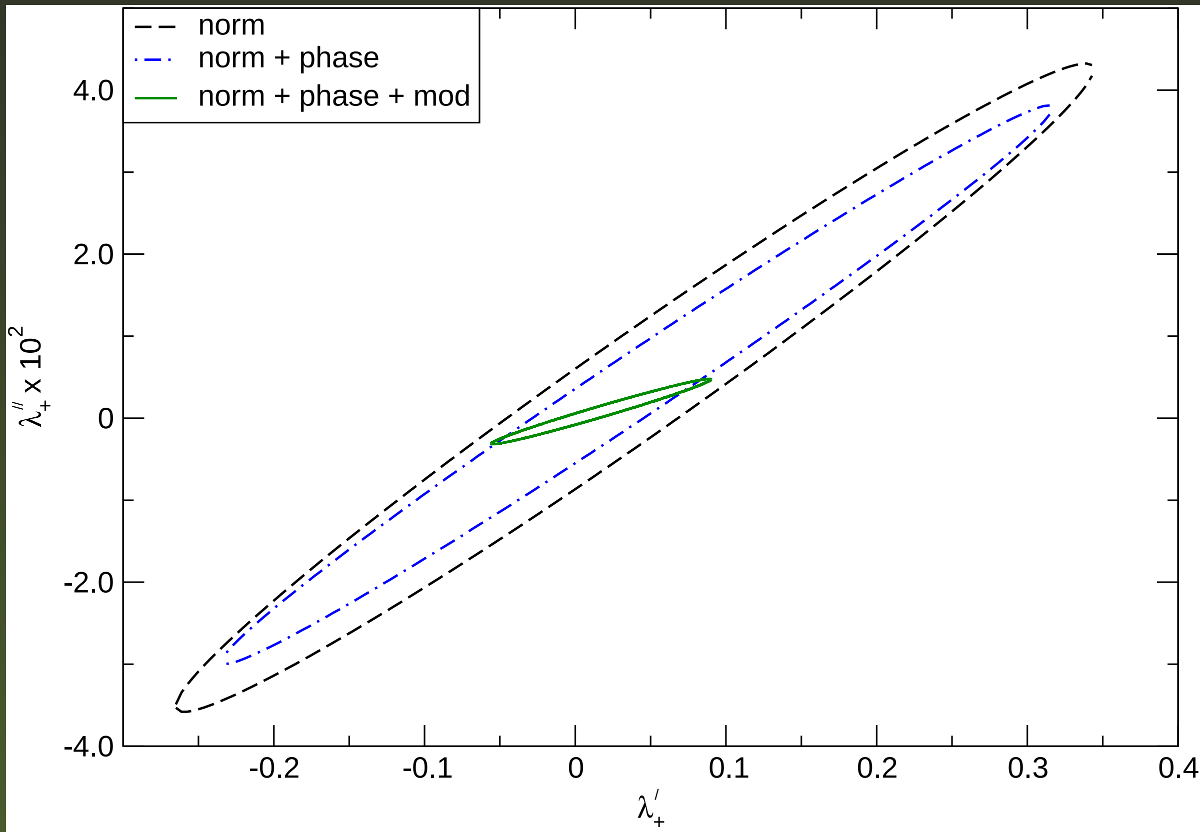
Best results for scalar shape parameters



Best results for scalar shape parameters with CT



Results for vector shape parameters



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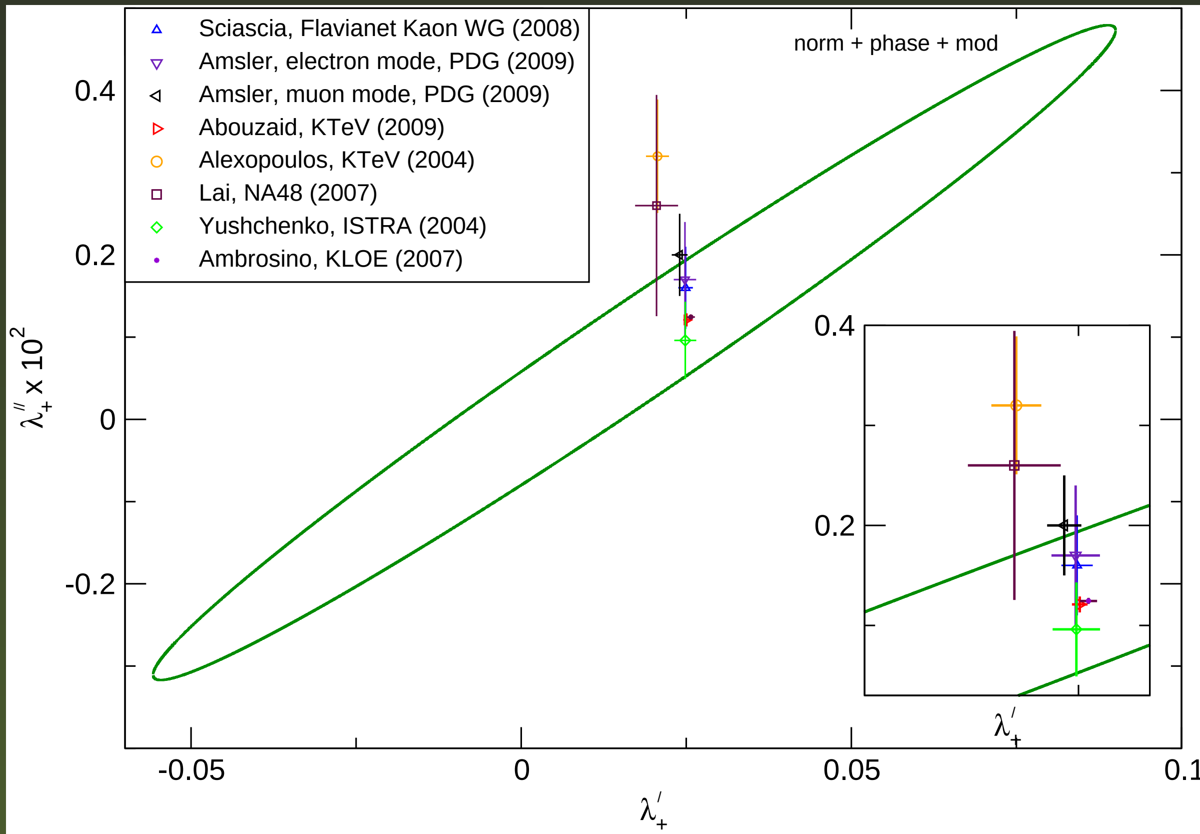

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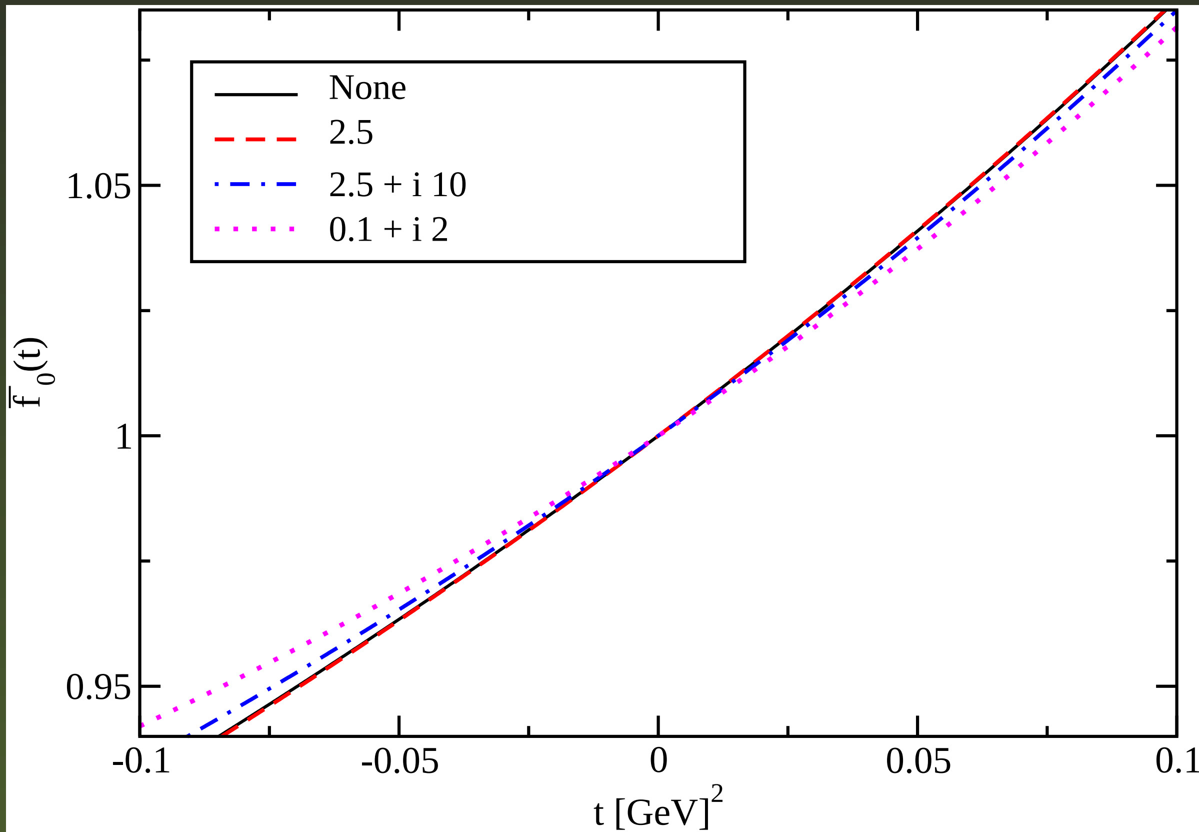
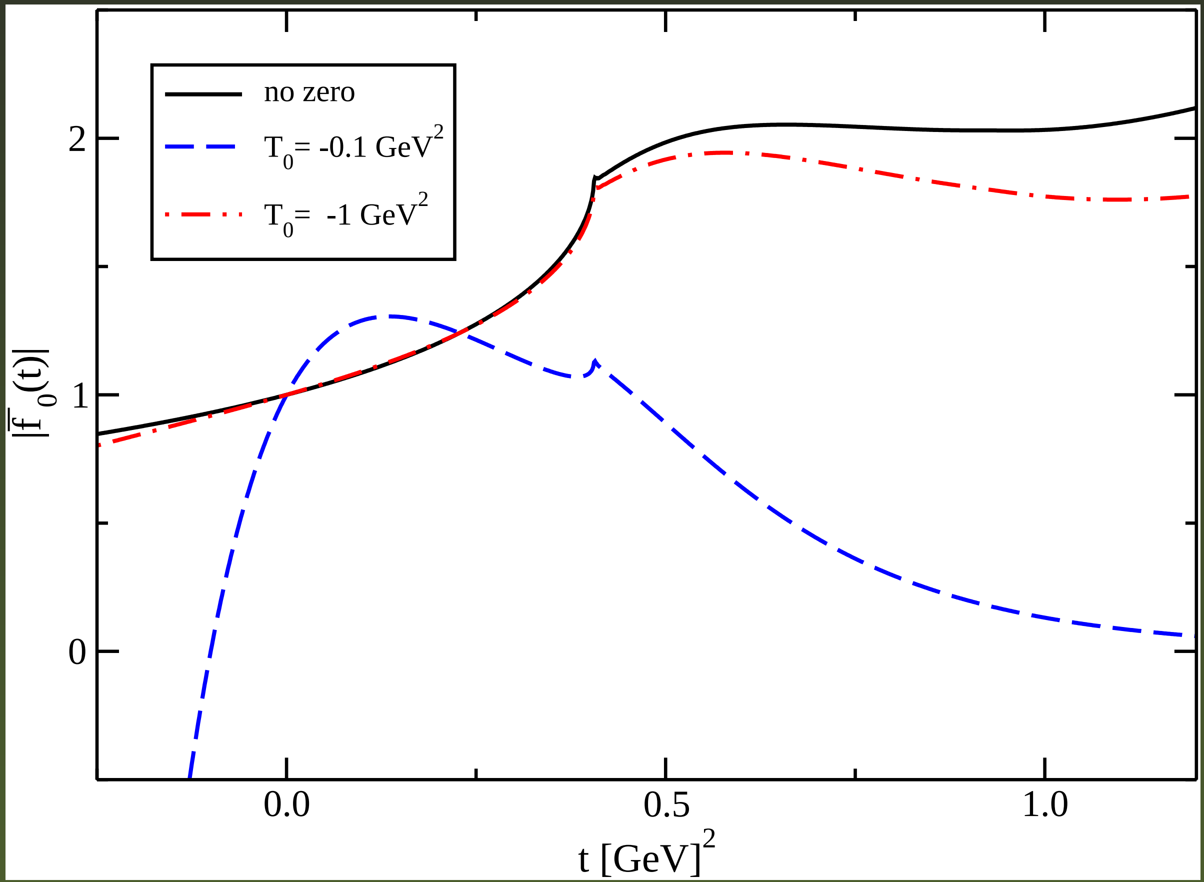
- Our method allows us search for zeros by using it as a SL constraint for both real and complex zeros

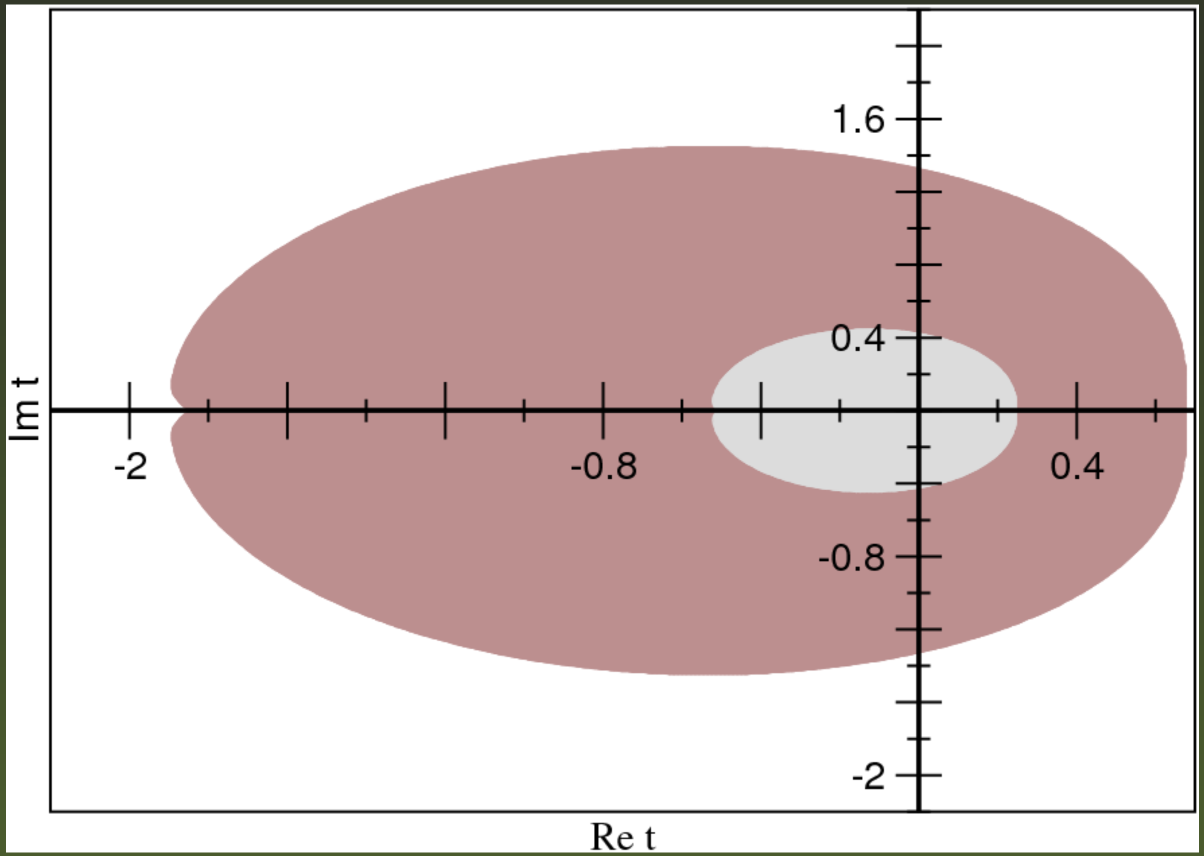
Influence of timelike zeros



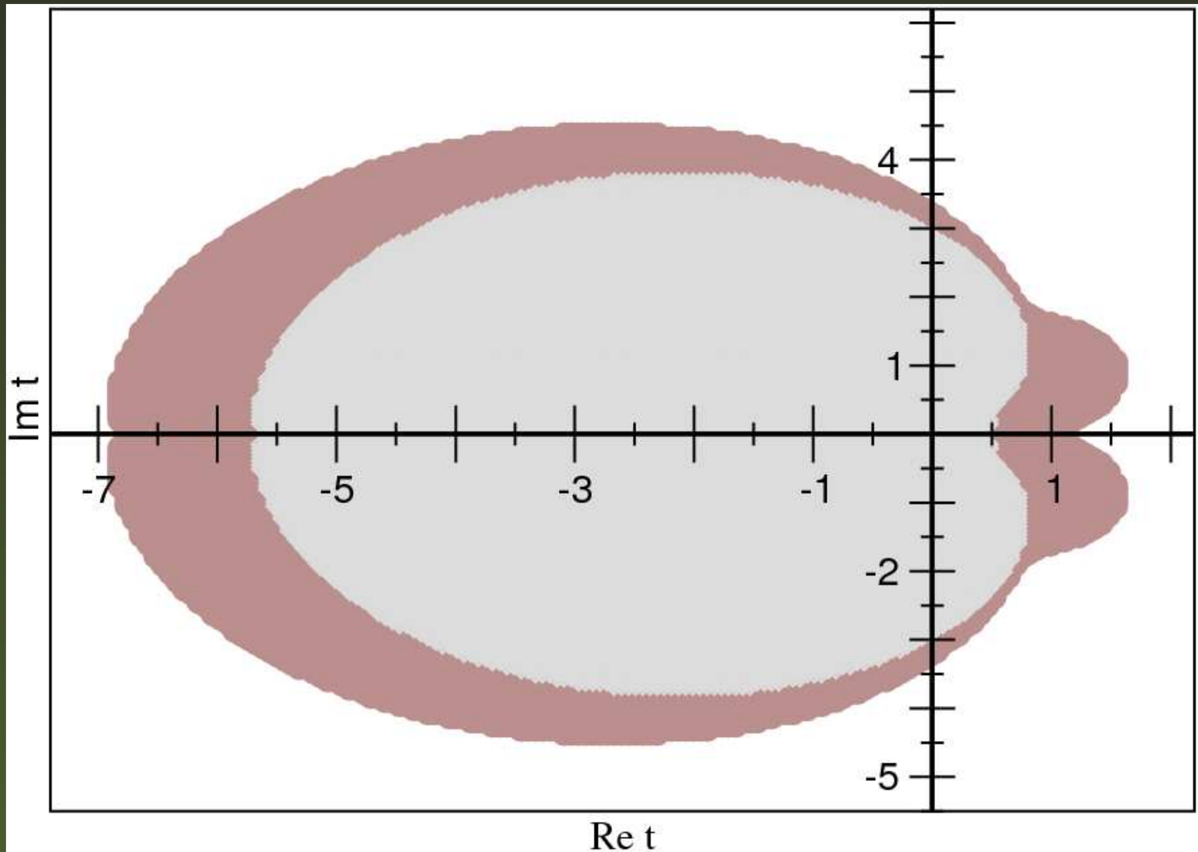
Influence of spacelike zeros



Absence of zeros for the vector



Absence of zeros for the scalar including CT



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